

# Second-order power spectra of CMB anisotropies due to primordial random perturbations in flat cosmological models

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(Dated: June 5, 2008)

Second-order power spectra of Cosmic Microwave Background (CMB) anisotropies due to random primordial perturbations at the matter dominant stage are studied, based on the relativistic second-order theory of perturbations in flat cosmological models and on the second-order formula of CMB anisotropies derived by Mollerach and Matarrese. So far the second-order integrated Sachs-Wolfe effect has been analyzed using the three-point correlation or bispectrum. In this paper we derive the second-order term of power spectra given using the two-point correlation of temperature fluctuations.

The second-order density perturbations are small, compared with the first-order ones. The second-order power spectra of CMB anisotropies, however, are not small at all, compared with the first-order power spectra, because at the early stage the first-order integrated Sachs-Wolfe effect is very small and the second-order integrated Sachs-Wolfe effect may be dominant over the first-order ones. So their characteristic behaviors may be measured through the future precise observation and bring useful informations on the structure and evolution of our universe in the future.

PACS numbers: 98.80.-k, 98.70.Vc, 04.25.Nx

## I. INTRODUCTION

In most studies of Cosmic Microwave Background (CMB) anisotropies, the comparison between observed and theoretical quantities have so far been done, assuming the linear approximation for cosmological perturbations. It seems to be successful enough to determine the cosmological parameters [1, 2, 3]. The present state of our universe is, however, locally complicated and associated with nonlinear behavior on various scales, and so the observed quantities of CMB anisotropies may include some small effects caused through nonlinear process by various primordial perturbations. So far the nonlinearity in CMB anisotropies has been treated as the second-order integrated Sachs-Wolfe effect, which was analyzed using the three-point correlation and bispectrum.[4, 5, 6] Recently a general treatment of second-order temperature fluctuations has been systematically studied by Bartolo, Matarrese and Riotto[7, 8] due to the transfer function which was derived from the Boltzmann equation, so as to be applied not only to the integrated Sachs-Wolfe effect at the matter-dominant stage, but also the nonlinear effect at the recombination epoch and the primordial nonlinear effect.

In recent years, on the other hand, we have studied these nonlinear effects of inhomogeneities on CMB anisotropies at the matter-dominant stage, based on the relativistic second-order theory of cosmological perturbations,[9] and on Mollerach and Matarrese's second-order formula of CMB anisotropies[10]. In previous papers,[11, 12, 13, 14, 15] we have studied the second-order effects of a local special large-scale inhomogeneity on CMB anisotropies, paying attention to the interaction between it and primordial random perturbations. In this relativistic theory we used the first-order and second-order perturbations in pressureless matter which were derived in compact and analytic forms. In the present paper we study the nonlinear effect of only primordial random perturbations on CMB anisotropies and derive the second-order power spectra, which is given as the second term in the two-point correlation of CMB anisotropies. This is a nonlinear correction to the first-order power spectra and different from the three-point correlation and bispectrum which have been used in the above other works to investigate uniquely the non-Gaussianity in the perturbations generated in various stages.

The second-order random density perturbations themselves are small, compared with the first-order ones. In the  $\Lambda$ -dominated model, however, the second-order power spectra of CMB anisotropies are not small at all, compared with the first-order power spectra, because at the early stage the first-order integrated Sachs-Wolfe effect (ISW) is very small and the second-order ISW may be dominant over the first-order one. Especially in the local void model with  $\Lambda = 0$ [18, 19, 20, 21], the exterior region is described by use of the Einstein-de Sitter model, in which the first-order ISW vanishes and the second-order ISW is indispensable. So their characteristic behaviors may be measured through the future precise observation and bring useful informations on the structure and evolution of our universe in the future.

In Sec. II, we show the second-order perturbations in general flat cosmological models and the corresponding CMB anisotropies. In Sec. III, we derive the expressions for the second-order power spectra of CMB anisotropies, due to primordial random perturbations. Concluding remarks follow in Section IV.

## II. SECOND-ORDER METRIC PERTURBATIONS AND TEMPERATURE ANISOTROPIES

First we review the background spacetime and the perturbations which were derived in the previous paper[11]. The background flat model with dust matter is expressed as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = a^2(\eta)[-d\eta^2 + \delta_{ij} dx^i dx^j], \quad (2.1)$$

where the Greek and Latin letters denote 0, 1, 2, 3 and 1, 2, 3, respectively, and  $\delta_{ij} (= \delta_j^i = \delta^{ij})$  are the Kronecker delta. The conformal time  $\eta (= x^0)$  is related to the cosmic time  $t$  by  $dt = a(\eta)d\eta$ . The matter density  $\rho$  and the scale factor  $a$  have the relations

$$\rho a^2 = 3(a'/a)^2 - \Lambda a^2, \quad \text{and} \quad \rho a^3 = \rho_0, \quad (2.2)$$

where a prime denotes  $\partial/\partial\eta$ ,  $\Lambda$  is the cosmological constant,  $\rho_0$  is an integration constant and the units  $8\pi G = c = 1$  are used.

The first-order and second-order metric perturbations  $\delta_1 g_{\mu\nu} (\equiv h_{\mu\nu})$  and  $\delta_2 g_{\mu\nu} (\equiv \ell_{\mu\nu})$ , respectively, were derived explicitly by imposing the synchronous coordinate condition:

$$h_{00} = h_{0i} = 0 \quad \text{and} \quad \ell_{00} = \ell_{0i} = 0. \quad (2.3)$$

Here we show their expressions only in the growing mode:

$$\begin{aligned} h_i^j &= P(\eta) F_{,ij}, \\ \ell_i^j &= P(\eta) L_i^j + P^2(\eta) M_i^j + Q(\eta) N_{|i}^{[j} + C_i^j, \end{aligned} \quad (2.4)$$

where  $F$  is an arbitrary potential function of spatial coordinates  $x^1, x^2$  and  $x^3$ ,  $h_i^j = \delta^{jl} h_{li}$ ,  $N_{|i}^{[j} = \delta^{jl} N_{|li} = N_{,ij}$ ,  $F_{,ij}$  means  $\partial^2 F / \partial x^i \partial x^j$ , and  $P(\eta)$  and  $Q(\eta)$  satisfy

$$\begin{aligned} P'' + \frac{2a'}{a} P' - 1 &= 0, \\ Q'' + \frac{2a'}{a} Q' &= -\left[P - \frac{5}{2}(P')^2\right]. \end{aligned} \quad (2.5)$$

The three-dimensional covariant derivative  $|i$  are defined in the space with metric  $dl^2 = \delta_{ij} dx^i dx^j$  and their suffixes are raised and lowered by use of  $\delta_{ij}$ . The functions  $L_i^j$  and  $M_i^j$  are defined by

$$\begin{aligned} L_i^j &= \frac{1}{2} \left[ -3F_{,i} F_{,j} - 2F \cdot F_{,ij} + \frac{1}{2} \delta_{ij} F_{,l} F_{,l} \right], \\ M_i^j &= \frac{1}{28} \left\{ 19F_{,il} F_{,jl} - 12F_{,ij} \Delta F - 3\delta_{ij} [F_{,kl} F_{,kl} - (\Delta F)^2] \right\} \end{aligned} \quad (2.6)$$

and  $N$  is defined by

$$\Delta N = \frac{1}{28} [(\Delta F)^2 - F_{,kl} F_{,kl}]. \quad (2.7)$$

The last term  $C_i^l$  satisfies the wave equation

$$\square C_i^j = \frac{3}{14} (P/a)^2 G_i^j + \frac{1}{7} \left[ P - \frac{5}{2} (P')^2 \right] \tilde{G}_i^j, \quad (2.8)$$

where  $G_i^j$  and  $\tilde{G}_i^j$  are second-order traceless and transverse functions of spatial coordinates, and the operator  $\square$  is defined by

$$\square \phi \equiv g^{\mu\nu} \phi_{;\mu\nu} = -a^{-2} \left( \partial^2 / \partial \eta^2 + \frac{2a'}{a} \partial / \partial \eta - \Delta \right) \phi, \quad (2.9)$$

where  $\square$  denotes the four-dimensional covariant derivative. So  $C_i^j$  represents the second-order gravitational waves caused by the first-order density perturbations.

The velocity perturbations  $\delta_1 u^\mu$  and  $\delta_2 u^\mu$  vanish, i.e.  $\delta_1 u^0 = \delta_1 u^i = 0$  and  $\delta_2 u^0 = \delta_2 u^i = 0$ , and the density perturbations are

$$\begin{aligned}\delta_1 \rho / \rho &= \frac{1}{\rho a^2} \left( \frac{a'}{a} P' - 1 \right) \Delta F, \\ \delta_2 \rho / \rho &= \frac{1}{2 \rho a^2} \left\{ \frac{1}{2} \left( 1 - \frac{a'}{a} P' \right) (3 F_{,l} F_{,l} + 8 F \Delta F) + \frac{1}{2} P [(\Delta F)^2 + F_{,kl} F_{,kl}] \right. \\ &\quad \left. + \frac{1}{4} [(P')^2 - \frac{2}{7} \frac{a'}{a} Q'] [(\Delta F)^2 - F_{,kl} F_{,kl}] - \frac{1}{7} \frac{a'}{a} P P' [4 F_{,kl} F_{,kl} + 3 (\Delta F)^2] \right\}.\end{aligned}\quad (2.10)$$

Next let us consider the CMB temperature  $T = T^{(0)}(1 + \delta T/T)$ , in which  $T^{(0)}$  is the background temperature and  $\delta T/T (= \delta_1 T/T + \delta_2 T/T)$  is the perturbations. The present temperature  $T_o^{(0)}$  is related to the emitted background temperature  $T_e^{(0)}$  at the recombination epoch by  $T_e^{(0)} = (1 + z_e) T_o^{(0)}$ , the temperature perturbation  $\tau \equiv (\delta T/T)_e$  at the recombination epoch is determined by the physical state before that epoch, and the present temperature perturbations  $(\delta T/T)_o$  is related to  $(\delta T/T)_e$  by the gravitational perturbations along the light ray from the recombination epoch to the present epoch. The light ray in the unperturbed state is described using the background wave vector  $k^\mu$  ( $\equiv dx^\mu/d\lambda$ ), where  $\lambda$  is the affine parameter, and its component is  $k^{(0)\mu} = (1, -e^i)$ , and the ray is given by  $x^{(0)\mu} = [\lambda, (\lambda_0 - \lambda)e^i]$ , where  $e^i$  is the directional unit vector. Here and in the following the suffixes  $o$  and  $e$  for  $\lambda, \eta$  and  $r$  denote the present (observed) epoch and the recombination (emitted) epoch, respectively.

The first-order temperature perturbation is

$$\delta_1 T/T = \tau + \frac{1}{2} \int_{\lambda_o}^{\lambda_e} d\lambda P'(\eta) F_{,ij} e^i e^j. \quad (2.11)$$

Using the relations  $dP/d\lambda = P'$  and  $dF/d\lambda = -F_{,i} e^i$ , this equation (2.11) is expressed as

$$\delta_1 T/T = \Theta_1 + \Theta_2 \quad (2.12)$$

where

$$\begin{aligned}\Theta_1 &\equiv \tau - \frac{1}{2} [(P' F_{,i})_e - (P' F_{,i})_o] e^i, \\ \Theta_2 &\equiv \frac{1}{2} \int_{\lambda_o}^{\lambda_e} d\lambda P''(\eta) F_{,i} e^i.\end{aligned}\quad (2.13)$$

$\Theta_1$  and  $\Theta_2$  represent the intrinsic and Sachs-Wolfe effects, respectively. The latter can be divided into the ordinary Sachs-Wolfe effect  $\Theta_{sac}$  and the Integrated Sachs-Wolfe effect  $\Theta_{isw}$ , where

$$\begin{aligned}\Theta_{sac} &\equiv \frac{1}{2} [(P'' F)_e - (P'' F)_o], \\ \Theta_{isw} &\equiv \frac{1}{2} \int_{\lambda_o}^{\lambda_e} d\lambda P'''(\eta) F.\end{aligned}\quad (2.14)$$

The second-order temperature perturbation is

$$\begin{aligned}\delta_2 T/T &= I_1(\lambda_e) \left[ \frac{1}{2} I_1(\lambda_e) - \tau \right] - [A_e^{(1)'} + \tau_{,i} e^i] \int_{\lambda_o}^{\lambda_e} d\lambda A^{(1)} \\ &\quad - \int_{\lambda_o}^{\lambda_e} d\lambda \left\{ \frac{1}{2} A^{(2)'} + A^{(1)} A^{(1)'} - A^{(1)''} \int_{\lambda_o}^{\lambda} d\bar{\lambda} A^{(1)}(\bar{\lambda}) \right\} + \frac{\partial \tau}{\partial d^i} d^{(1)i},\end{aligned}\quad (2.15)$$

where  $(\eta, x^i) = (\lambda, \lambda_o - \lambda)$  in the integrands and

$$\begin{aligned}I_1(\lambda_e) &= -\frac{1}{2} \int_{\lambda_o}^{\lambda_e} d\lambda P' F_{,ij} e^i e^j, \\ A^{(1)} &= -\frac{1}{2} P F_{,ij} e^i e^j, \\ A^{(2)} &= -\frac{1}{2} [P L_i^j + P^2 M_i^j + Q N_{,ij} + C_i^j] e^i e^j.\end{aligned}\quad (2.16)$$

These expressions were derived in §3 of a previous paper.[9]

### III. POWER SPECTRA OF SECOND-ORDER CMB ANISOTROPIES

We consider primordial scalar perturbations with  $F$  defined by

$$F = \int d\mathbf{k} \alpha(\mathbf{k}) e^{i\mathbf{k}\mathbf{x}}, \quad (3.1)$$

where the spatial average for  $\alpha(\mathbf{k})$  is given by

$$\langle \alpha(\mathbf{k}) \alpha(\bar{\mathbf{k}}) \rangle = (2\pi)^{-2} \mathcal{P}_F(\mathbf{k}) \delta(\mathbf{k} + \bar{\mathbf{k}}), \quad (3.2)$$

with

$$\mathcal{P}_F(\mathbf{k}) = \mathcal{P}_{F0} k^{-3} (k/k_0)^{n-1} T_s^2(k), \quad (3.3)$$

where  $T_s(k)$  is the matter transfer function [22] and  $\mathcal{P}_{F0}$  is the normalization constant.

Then the first-order temperature perturbations are

$$\delta_1 T/T \equiv \Theta_P = -\frac{1}{2} \int d\mathbf{k} \alpha(\mathbf{k}) \int_{\lambda_o}^{\lambda_e} d\lambda P'(\eta) (k\mu)^2 e^{i\mathbf{k}\mathbf{x}}, \quad (3.4)$$

where  $\mathbf{x} = r\mathbf{e}$ ,  $\mathbf{k}\mathbf{x} = kr\mu$ ,  $\mu \equiv \cos \theta_k$  and  $\theta_k$  is the angle between the wave vector  $k^i$  and a unit vector  $e^i$ . This equation can be rewritten as

$$\Theta_P = \int d\mathbf{k} \alpha(\mathbf{k}) \left\{ -\frac{1}{2} [(P'')_o + ik(P')_o P_1(\mu)] + \frac{1}{2} \sum_l (-i)^l (2l+1) \Theta_{P(l)} P_l(\mu) \right\}, \quad (3.5)$$

where  $P_l(\mu)$  is the Legendre polynomial and

$$\Theta_{P(l)} = \int_{\lambda_o}^{\lambda_e} d\lambda P'''(\eta) j_l(kr) - \{k(P')_e (2l+1)^{-1} [(l+1)j_{l+1}(kr_e) - lj_{l-1}(kr_e)] + j_l(kr_e) (P'')_e\}. \quad (3.6)$$

In these equations, we have  $\eta = \lambda$  and  $r = \lambda_o - \lambda$ . In the derivation of Eq.(3.6), we used the relations[23, 24]

$$e^{i\mathbf{k}\mathbf{x}} = e^{ikr\mu} = \sum_l (-i)^l (2l+1) j_l(kr) P_l(\mu) \quad (3.7)$$

and

$$(2l+1)\mu P_l(\mu) = (l+1)P_{l+1}(\mu) + lP_{l-1}(\mu). \quad (3.8)$$

The components of the unit vector  $e^i$  are expressed as  $e^1 = \sin \theta \cos \phi$ ,  $e^2 = \sin \theta \sin \phi$ ,  $e^3 = \cos \theta$  with respect to  $x^1, x^2, x^3$  axes, respectively.

In order to derive the power spectra, we take the statistical average  $\langle \rangle$  for the primordial perturbations, and  $\langle (\delta T/T)^2 \rangle$  is expressed for the first-order anisotropies as

$$\langle (\delta_1 T/T)^2 \rangle = \langle (\Theta_P)^2 \rangle = (T_0)^{-2} \sum_l \frac{2l+1}{4\pi} C_l. \quad (3.9)$$

The power spectra  $C_l$  are

$$C_l = (T_0)^2 \int dk k^2 \mathcal{P}_F(k) |\mathcal{H}_P^{(l)}(k)|^2, \quad (3.10)$$

where  $T_0$  is the present CMB temperature and

$$\begin{aligned} \mathcal{H}_P^{(0)}(k) &= -(P'')_o - k(P')_e j_1(kr_e) - (P'')_e j_0(kr_e) + \int_{\lambda_o}^{\lambda_e} d\lambda P''' j_0(kr), \\ \mathcal{H}_P^{(1)}(k) &= \frac{1}{3} k [(P')_o - (P')_e] j_1^{(1)}(kr_e) - (P'')_e j_1(kr_e) + \int_{\lambda_o}^{\lambda_e} d\lambda P''' j_1(kr). \end{aligned} \quad (3.11)$$

For  $l \geq 2$ , we have

$$\mathcal{H}_P^{(l)}(k) = k(P')_e j_l^{(1)}(kr_e) - (P'')_e j_l(kr_e) + \int_{\lambda_o}^{\lambda_e} d\lambda P''' j_l(kr). \quad (3.12)$$

In the derivation of  $C_l$ , we used Eq.(3.2) for  $\langle \alpha(\mathbf{k})\alpha(\bar{\mathbf{k}}) \rangle$  and a mathematical formula for  $P_l(\mu)$ :

$$\int_{-1}^1 P_l(\mu) P_{l'}(\mu) d\mu = 2/(2l+1), \quad 0 \quad \text{for} \quad l = l', \quad l \neq l', \quad (3.13)$$

respectively. For the two directions with unit vectors  $\mathbf{e}_1$  and  $\mathbf{e}_2$ , we have the correlation

$$(T_0)^2 \langle \Theta_P(\mathbf{e}_1) \Theta_P(\mathbf{e}_2) \rangle = \sum_l \frac{2l+1}{4\pi} C_l P_l(\cos \beta). \quad (3.14)$$

where the product  $\mathbf{e}_1 \mathbf{e}_2$  is equal to  $\cos \beta$ .

For the second-order temperature anisotropies, we obtain from Eqs.(2.15), (2.16) and (3.1)

$$\begin{aligned} \delta_2 T/T &= \int \int d\mathbf{k} d\bar{\mathbf{k}} \alpha(\mathbf{k}) \alpha(\bar{\mathbf{k}}) \left\{ \frac{1}{8} \int_{\lambda_o}^{\lambda_e} \int_{\lambda_o}^{\lambda_e} d\lambda d\bar{\lambda} P'(\lambda) P'(\bar{\lambda}) (k_e \bar{k}_e)^2 e^{i(\mathbf{k}\mathbf{x} + \bar{\mathbf{k}}\bar{\mathbf{x}})} \right. \\ &+ \int_{\lambda_o}^{\lambda_e} d\lambda \left[ \frac{1}{8} P'(\lambda) \left( 3k_e \bar{k}_e + (k_e)^2 + (\bar{k}_e)^2 - \frac{1}{2} \mathbf{k}\bar{\mathbf{k}} \right) \right. \\ &+ \frac{1}{56} P(\lambda) P'(\lambda) \left( 19k_e \bar{k}_e \mathbf{k}\bar{\mathbf{k}} - 14(k_e \bar{k}_e)^2 - 6(\bar{k}_e k)^2 - 6(k_e \bar{k})^2 - 3(\mathbf{k}\bar{\mathbf{k}})^2 + 3k^2 (\bar{k})^2 \right) \\ &+ \left. \frac{1}{112} Q'(\lambda) (k_e + \bar{k}_e)^2 \left( k^2 \bar{k}^2 - (\mathbf{k}\bar{\mathbf{k}})^2 \right) / (\mathbf{k} + \bar{\mathbf{k}})^2 \right] e^{i(\mathbf{k} + \bar{\mathbf{k}})\mathbf{x}} \\ &+ \left. \frac{1}{4} \int_{\lambda_o}^{\lambda_e} d\lambda \left[ P''(\lambda) \int_{\lambda_o}^{\lambda} d\bar{\lambda} P(\bar{\lambda}) (k_e \bar{k}_e)^2 \right] e^{i(\mathbf{k}\mathbf{x} + \bar{\mathbf{k}}\bar{\mathbf{x}})} \right\}, \end{aligned} \quad (3.15)$$

where  $k_e$  stands for  $\mathbf{k}\mathbf{e}$ ,  $P'(\lambda) \equiv dP(\lambda)/d\lambda$ , and  $P'(\bar{\lambda}) \equiv dP(\bar{\lambda})/d\bar{\lambda}$ . Here  $\tau$  and  $A_e^{(1)'} in Eq.(2.15) were neglected, because we pay attentions to the Sachs-Wolfe effect after the recombination epoch, and also the term with  $C_i^j$  in Eq.(2.16) was neglected, because the contribution of gravitational radiation is very small. It is found from the above equation that the average  $\langle \delta_2 T/T \rangle$  does not vanish in contrast to the vanishing first-order one ( $\langle \delta_1 T/T \rangle$ ), and we obtain$

$$\begin{aligned} \langle \delta_2 T/T \rangle &= \frac{1}{8} \int d\mathbf{k} (2\pi)^{-2} \mathcal{P}_F(\mathbf{k}) \left\{ \int_{\lambda_o}^{\lambda_e} \int_{\lambda_o}^{\lambda_e} d\lambda d\bar{\lambda} P'(\lambda) P'(\bar{\lambda}) (k_e)^4 e^{i\mathbf{k}(\mathbf{x} - \bar{\mathbf{x}})} \right. \\ &+ \int_{\lambda_o}^{\lambda_e} d\lambda \left[ P'(\lambda) \left( -(k_e)^2 + \frac{1}{2} k^2 \right) + P(\lambda) P'(\lambda) \left( k^2 - 2(k_e)^2 \right) (k_e)^2 \right] \\ &+ \left. 2 \int_{\lambda_o}^{\lambda_e} d\lambda P''(\lambda) \int_{\lambda_o}^{\lambda} d\bar{\lambda} P(\bar{\lambda}) (k_e)^4 e^{i\mathbf{k}(\mathbf{x} - \bar{\mathbf{x}})} \right\}, \end{aligned} \quad (3.16)$$

where  $k_e \equiv \mathbf{k}\mathbf{e} = k\mu$  and  $d\mathbf{k} = dk k^2 d\phi d\mu$ . Here we have the relations  $\mathbf{x} = r\mathbf{e}$ ,  $\bar{\mathbf{x}} = \bar{r}\mathbf{e}$ , and so  $\mathbf{k}\mathbf{x} = k_e r = kr\mu$ ,  $-\mathbf{k}\bar{\mathbf{x}} = k\bar{r}(-\mu)$ . Thus  $e^{i\mathbf{k}(\mathbf{x} - \bar{\mathbf{x}})}$  is expanded as

$$\begin{aligned} e^{i\mathbf{k}(\mathbf{x} - \bar{\mathbf{x}})} &= \sum_l (-i)^l (2l+1) j_l(kr) P_l(\mu) \times \sum_{l'} (-i)^{l'} (2l'+1) j_{l'}(k\bar{r}) P_{l'}(-\mu) \\ &= \sum_{l,l'} (-i)^l i^{l'} (2l+1)(2l'+1) j_l(kr) j_{l'}(k\bar{r}) P_l(\mu) P_{l'}(\mu). \end{aligned} \quad (3.17)$$

Here for the reduction of Eq.(3.16), we derive the following relations using Eq.(3.8)

$$\sum_l j_l \mu P_l = \sum_l j_l^{(1)} P_l, \quad \sum_l j_l \mu^2 P_l = \sum_l j_l^{(2)} P_l, \quad \text{and} \quad \sum_l j_l \mu^3 P_l = \sum_l j_l^{(3)} P_l, \quad (3.18)$$

where

$$j_l^{(1)}(kr) \equiv \frac{l}{2l-1} j_{l-1}(kr) + \frac{l+1}{2l+3} j_{l+1}(kr),$$

$$\begin{aligned}
j_l^{(2)}(kr) &\equiv \frac{(l-1)l}{(2l-3)(2l-1)}j_{l-2}(kr) + \frac{1}{2l+1}\left[\frac{l^2}{2l-1} + \frac{(l+1)^2}{(2l+3)}\right]j_l(kr) + \frac{(l+1)(l+2)}{(2l+3)(2l+5)}j_{l+2}(kr), \\
j_l^{(3)}(kr) &\equiv \frac{(l-2)(l-1)l}{(2l-5)(2l-3)(2l-1)}j_{l-3}(kr) + \frac{l}{2l-1}\left[\frac{(l-1)^2}{(2l-3)(2l-1)}\right. \\
&\quad \left. + \frac{l^2}{(2l-1)(2l+1)} + \frac{(l+1)^2}{(2l+1)(2l+3)}\right]j_{l-1}(kr) + \frac{l+1}{2l+3}\left[\frac{l^2}{(2l-1)(2l+1)}\right. \\
&\quad \left. + \frac{(l+1)^2}{(2l+1)(2l+3)} + \frac{(l+2)^2}{(2l+3)(2l+5)}\right]j_{l+1}(kr) + \frac{(l+1)(l+2)(l+3)}{(2l+3)(2l+5)(2l+7)}j_{l+3}(kr).
\end{aligned} \tag{3.19}$$

Then we have

$$\mu^4 e^{i\mathbf{k}(\mathbf{x}-\bar{\mathbf{x}})} = \sum_{l,l'} (-i)^l i^{l'} (2l+1)(2l'+1) j_l^{(2)}(kr) j_{l'}^{(2)}(k\bar{r}) P_l(\mu) P_{l'}(\mu). \tag{3.20}$$

Using these relations and executing the integrations in Eq.(3.16) with respect to  $\phi$  and  $\mu$ , we obtain

$$\begin{aligned}
\langle \delta_2 T/T \rangle &= (2\pi)^{-1} \int dk k^2 \mathcal{P}_F(\mathbf{k}) \left\{ \frac{1}{4} k^4 \sum_l (2l+1) \left[ \int_{\lambda_o}^{\lambda_e} d\lambda P'(\lambda) j_l^{(2)}(kr) \right]^2 \right. \\
&\quad \left. + \frac{1}{24} (P_e - P_o) k^2 + \frac{17}{840} (P_e^2 - P_o^2) k^4 \right. \\
&\quad \left. + \frac{1}{2} k^4 \sum_l (2l+1) \int_{\lambda_o}^{\lambda_e} d\lambda P''(\lambda) j_l^{(2)}(kr) \int_{\lambda_o}^{\lambda} d\bar{\lambda} P(\bar{\lambda}) j_l^{(2)}(k\bar{r}) \right\},
\end{aligned} \tag{3.21}$$

where we used the relation (3.13), and  $P_e = P(\lambda_e)$  and  $P_o = P(\lambda_o)$ .

When we take into account also the third-order anisotropies  $\delta_3 T/T$ , we have

$$\begin{aligned}
\delta T/T &= \delta_1 T/T + \delta_2 T/T + \delta_3 T/T \\
&= \delta_1 T/T + \langle \delta_2 T/T \rangle + \Theta_{pp} + \delta_3 T/T,
\end{aligned} \tag{3.22}$$

where  $\Theta_{pp} \equiv \delta_2 T/T - \langle \delta_2 T/T \rangle$ . Then the total average of  $(\delta T/T)^2$  is expressed as

$$\langle (\delta T/T)^2 \rangle = \langle \Theta_{pp}^2 \rangle + (\langle \delta_2 T/T \rangle)^2 + \langle \Theta_{pp}^2 \rangle + 2 \langle \delta_2 T/T \delta_3 T/T \rangle. \tag{3.23}$$

The last term is of the same order as the second and third terms. At present we have not obtained any concrete expression of third-order metric perturbations yet and so the last term is not analyzed, while the formula for  $\delta_3 T/T$  and formal solutions have recently been derived by D'Amico et al.[16] and the perturbative equations to the third order were studied by Hwang and Noh[17] with respect to the relativistic-Newtonian correspondence. Here  $\langle \delta_2 T/T \rangle$  is the monopole component without angular dependence and  $\Theta_{pp}$  is the renormalized second-order temperature fluctuation, which have been discussed by Munshi et al.[5]

Now let us make a reduction of  $\langle \Theta_{pp}^2 \rangle$ . It is expressed as

$$\langle \Theta_{pp}^2 \rangle = \left\langle \left( \delta_2 T/T(\alpha(\mathbf{k})\alpha(\bar{\mathbf{k}})) - \langle \delta_2 T/T \rangle \right) \left( \delta_2 T/T(\alpha(\bar{\bar{\mathbf{k}}})\alpha(\bar{\bar{\mathbf{k}}})) - \langle \delta_2 T/T \rangle \right) \right\rangle, \tag{3.24}$$

where  $\delta_2 T/T(\alpha(\mathbf{k})\alpha(\bar{\mathbf{k}}))$  is given by Eq.(3.15) and  $\delta_2 T/T(\alpha(\bar{\bar{\mathbf{k}}})\alpha(\bar{\bar{\mathbf{k}}}))$  is obtained from Eq.(3.15) by replacing  $\mathbf{k}, \bar{\mathbf{k}}$  by  $\bar{\bar{\mathbf{k}}}, \bar{\bar{\mathbf{k}}}$ . The averaging process in Eq.(3.24) is performed in the two sets:  $\langle \alpha(\mathbf{k})\alpha(\bar{\mathbf{k}}) \rangle, \langle \alpha(\bar{\mathbf{k}})\alpha(\bar{\bar{\mathbf{k}}}) \rangle$  and  $\langle \alpha(\mathbf{k})\alpha(\bar{\bar{\mathbf{k}}}) \rangle, \langle \alpha(\bar{\mathbf{k}})\alpha(\bar{\mathbf{k}}) \rangle$ , and the average process for  $\langle \alpha(\mathbf{k})\alpha(\bar{\mathbf{k}}) \rangle$  and  $\langle \alpha(\bar{\mathbf{k}})\alpha(\bar{\bar{\mathbf{k}}}) \rangle$  is excluded by subtracting  $\langle \delta_2 T/T \rangle$  from  $\delta_2 T/T(\alpha(\mathbf{k})\alpha(\bar{\mathbf{k}}))$  and  $\delta_2 T/T(\alpha(\bar{\mathbf{k}})\alpha(\bar{\bar{\mathbf{k}}}))$ . When we sum these average values in the above two sets and symmetrize the expression with respect to  $\mathbf{k}$  and  $\bar{\mathbf{k}}$ , we obtain from Eq.(3.24)

$$\begin{aligned}
\langle \Theta_{pp}^2 \rangle &= 2 \int \int dk d\bar{k} (k\bar{k})^2 d\phi_k d\bar{\phi}_k d\mu d\bar{\mu} (2\pi)^{-4} \mathcal{P}_F(k) \mathcal{P}_F(\bar{k}) \\
&\quad \times \left\{ \frac{1}{8} \int_{\lambda_o}^{\lambda_e} \int_{\lambda_o}^{\lambda_e} d\lambda d\bar{\lambda} P'(\lambda) P'(\bar{\lambda}) (k_e \bar{k}_e)^2 e^{i(\mathbf{k}\mathbf{x} + \bar{\mathbf{k}}\bar{\mathbf{x}})} + \int_{\lambda_o}^{\lambda_e} d\lambda \left[ \frac{1}{8} P'(\lambda) (3k_e \bar{k}_e + (k_e)^2 + (\bar{k}_e)^2 - \frac{1}{2} \mathbf{k}\bar{\mathbf{k}}) \right. \right. \\
&\quad \left. \left. + \frac{1}{56} P(\lambda) P'(\lambda) (19k_e \bar{k}_e \mathbf{k}\bar{\mathbf{k}} - 14(k_e \bar{k}_e)^2 - 6(\bar{k}_e k)^2 - 6(k_e \bar{k})^2 - 3(\mathbf{k}\bar{\mathbf{k}})^2 + 3k^2 (\bar{k})^2) \right] e^{i(\mathbf{k} + \bar{\mathbf{k}})\mathbf{x}} \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} \int_{\lambda_o}^{\lambda_e} d\lambda P''(\lambda) \int_{\lambda_o}^{\lambda} P(\bar{\lambda}) (k_e \bar{k}_e)^2 e^{i(\mathbf{k}\mathbf{x} + \bar{\mathbf{k}}\bar{\mathbf{x}})} \Big\} \\
& \times \left\{ \frac{1}{8} \int_{\lambda_o}^{\lambda_e} \int_{\lambda_o}^{\lambda_e} d\eta d\bar{\eta} P'(\eta) P'(\bar{\eta}) (k_e \bar{k}_e)^2 e^{-i(\mathbf{k}\mathbf{y} + \bar{\mathbf{k}}\bar{\mathbf{y}})} + \int_{\lambda_o}^{\lambda_e} d\eta \left[ \frac{1}{8} P'(\eta) \left( 3k_e \bar{k}_e + (k_e)^2 + (\bar{k}_e)^2 - \frac{1}{2} \mathbf{k}\bar{\mathbf{k}} \right) \right. \right. \\
& + \frac{1}{56} P(\eta) P'(\eta) \left( 19k_e \bar{k}_e \mathbf{k}\bar{\mathbf{k}} - 14(k_e \bar{k}_e)^2 - 6(\bar{k}_e k_e)^2 - 6(k_e \bar{k})^2 - 3(\mathbf{k}\bar{\mathbf{k}})^2 + 3k^2(\bar{k})^2 \right) \Big] e^{-i(\mathbf{k} + \bar{\mathbf{k}})\mathbf{y}} \\
& \left. + \frac{1}{4} \int_{\lambda_o}^{\lambda_e} d\eta P''(\eta) \int_{\lambda_o}^{\eta} d\bar{\eta} P(\bar{\eta}) (k_e \bar{k}_e)^2 e^{-i(\mathbf{k}\mathbf{y} + \bar{\mathbf{k}}\bar{\mathbf{y}})} \right\}, \tag{3.25}
\end{aligned}$$

where  $\mathbf{x}, \bar{\mathbf{x}}$  and  $\mathbf{y}, \bar{\mathbf{y}}$  are functions of  $\lambda, \bar{\lambda}$  and  $\eta, \bar{\eta}$ , respectively, and we neglected the terms with  $Q$  and  $N_i^{(j)}$ , because we have  $Q/P^2 < 10^{-2}$  always and they are very small. Especially  $Q$  vanishes in the case  $\Lambda = 0$ . Here, when we consider an orthonormal triad vector  $e_{(1)}^i, e_{(2)}^i$ , and  $e_{(3)}^i (= e^i)$ , the components of  $\mathbf{k}$  and  $\bar{\mathbf{k}}$  with respect to this triad are expressed as  $\mathbf{k} = k(\sin \theta_k \cos \phi_k, \sin \theta_k \sin \phi_k, \cos \theta_k)$  and  $\bar{\mathbf{k}} = \bar{k}(\sin \bar{\theta}_k \cos \bar{\phi}_k, \sin \bar{\theta}_k \sin \bar{\phi}_k, \cos \bar{\theta}_k)$ , and so  $\mu = \cos \theta_k$  and  $\bar{\mu} = \cos \bar{\theta}_k$ .

Next let us take a notice of terms with  $\mathbf{k}\bar{\mathbf{k}}$  and  $(\mathbf{k}\bar{\mathbf{k}})^2$  which can be expressed as

$$\begin{aligned}
\mathbf{k}\bar{\mathbf{k}} &= k\bar{k} [\sin \theta_k \sin \bar{\theta}_k \cos(\phi_k - \bar{\phi}_k) + \cos \theta_k \cos \bar{\theta}_k], \\
(\mathbf{k}\bar{\mathbf{k}})^2 &= (k\bar{k})^2 \{ (\cos \theta_k \cos \bar{\theta}_k)^2 + \frac{1}{2} (\sin \theta_k \sin \bar{\theta}_k)^2 [1 + \cos 2(\phi_k - \bar{\phi}_k)] \\
&+ 2 \sin \theta_k \sin \bar{\theta}_k \cos \theta_k \cos \bar{\theta}_k \cos(\phi_k - \bar{\phi}_k) \}. \tag{3.26}
\end{aligned}$$

By integrations with respect to  $\phi_k$  and  $\bar{\phi}_k$ , we obtain  $\int \int d\phi_k d\bar{\phi}_k \cos(\phi_k - \bar{\phi}_k) = \int \int d\phi_k d\bar{\phi}_k \cos 2(\phi_k - \bar{\phi}_k) = 0$ , while  $\int \int \phi_k \bar{\phi}_k [\cos(\phi_k - \bar{\phi}_k)]^2 = \int \int d\phi_k d\bar{\phi}_k [\cos 2(\phi_k - \bar{\phi}_k)]^2 = (2\pi)^2/2$ .

Executing integrations in Eq.(3.25) with respect to  $\phi_k$  and  $\bar{\phi}_k$ , and using the relations Eq.(3.7) and

$$e^{-i\mathbf{k}\mathbf{y}} = e^{-iks\mu} = \sum_m (-i)^m (2m+1) j_m(ks) P_m(-\mu) = \sum_m i^m (2m+1) j_m(ks) P_m(\mu), \tag{3.27}$$

we can, therefore, reexpress Eq.(3.25) as

$$\langle (\Theta_{pp})^2 \rangle = \sum_l \sum_{l'} \sum_m \sum_{m'} \left( A_{ll'mm'}^I + A_{ll'mm'}^{II} + A_{ll'mm'}^{III} \right), \tag{3.28}$$

where the terms  $A_{ll'mm'}^I, A_{ll'mm'}^{II}$  and  $A_{ll'mm'}^{III}$  come from the terms without  $\phi_k$  and  $\bar{\phi}_k$ , the coefficients of  $\cos^2(\phi_k - \bar{\phi}_k)$ , and the coefficients of  $\cos^2 2(\phi_k - \bar{\phi}_k)$ , and their lengthy expressions are shown in Appendix A.

Moreover let us replace the terms of  $\mu j_l P_l$ ,  $\mu^2 j_l P_l$  and  $\mu^3 j_l P_l$  by  $j_l^{(1)} P_l$ ,  $j_l^{(2)} P_l$  and  $j_l^{(3)} P_l$  using Eqs. (3.18) and (3.19) and execute integrations with respect to  $\mu$  and  $\bar{\mu}$  using Eq.(3.13). Then we obtain

$$\langle (\Theta_{pp})^2 \rangle = \sum_l \sum_{l'} \left( B_{ll'}^I + B_{ll'}^{II} + B_{ll'}^{III} \right), \tag{3.29}$$

where

$$\begin{aligned}
B_{ll'}^I &= \frac{1}{8} (2\pi)^{-2} (2l+1)(2l'+1) \int \int dk d\bar{k} (k\bar{k})^2 \mathcal{P}_F(k) \mathcal{P}_F(\bar{k}) \\
&\times \left\{ (k\bar{k})^2 \int_{\lambda_o}^{\lambda_e} \int_{\lambda_o}^{\lambda_e} d\lambda d\bar{\lambda} P'(\lambda) P'(\bar{\lambda}) j_l^{(2)}(kr) j_{l'}^{(2)}(\bar{k}r) + 2(k\bar{k})^2 \int_{\lambda_o}^{\lambda_e} d\lambda P''(\lambda) \int_{\lambda_o}^{\lambda} d\bar{\lambda} P(\bar{\lambda}) j_l^{(2)}(kr) j_{l'}^{(2)}(\bar{k}r) \right. \\
&+ \int_{\lambda_o}^{\lambda_e} d\lambda \left[ P'(\lambda) \left( \frac{5}{2} k\bar{k} j_l^{(1)}(kr) j_{l'}^{(1)}(\bar{k}r) + k^2 j_l^{(2)}(kr) j_{l'}^{(2)}(\bar{k}r) + \bar{k}^2 j_l^{(2)}(kr) j_{l'}^{(2)}(\bar{k}r) \right) \right. \\
&+ \frac{3}{14} P(\lambda) P'(\lambda) (k\bar{k})^2 \left( j_l(kr) j_{l'}(\bar{k}r) - 3j_l^{(2)}(kr) j_{l'}^{(2)}(\bar{k}r) - 3j_l(kr) j_{l'}^{(2)}(\bar{k}r) \right. \\
&\left. \left. + \frac{1}{3} j_l^{(2)}(kr) j_{l'}^{(2)}(\bar{k}r) \right) \right] \Big\}^2, \tag{3.30}
\end{aligned}$$

$$B_{ll'}^{II} = \frac{1}{32} (2\pi)^{-2} (2l+1)(2l'+1) \int \int dk d\bar{k} (k\bar{k})^4 \mathcal{P}_F(k) \mathcal{P}_F(\bar{k})$$

$$\begin{aligned}
& \times \int_{\lambda_o}^{\lambda_e} d\lambda P'(\lambda) \left[ \left( j_l(kr) - j_l^{(2)}(kr) \right) - \frac{26}{7} k \bar{k} P(\lambda) \left( j_l^{(1)}(kr) - j_l^{(3)}(kr) \right) \right] j_{l'}(kr) \\
& \times \int_{\lambda_o}^{\lambda_e} d\eta P'(\eta) \left[ \left( j_{l'}(\bar{k}s) - j_{l'}^{(2)}(\bar{k}s) \right) - \frac{26}{7} k \bar{k} P(\eta) \left( j_{l'}^{(1)}(\bar{k}s) - j_{l'}^{(3)}(\bar{k}s) \right) \right] j_l(\bar{k}s),
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
B_{ll'}^{III} &= 2 \left( \frac{3}{56} \right)^2 (2\pi)^{-2} (2l+1)(2l'+1) \int \int dk d\bar{k} (k\bar{k})^6 \mathcal{P}_F(k) \mathcal{P}_F(\bar{k}) \\
& \times \left\{ \int_{\lambda_o}^{\lambda_e} d\lambda P(\lambda) P'(\lambda) \left[ j_l(kr) j_{l'}(\bar{k}r) - j_l^{(2)}(kr) j_{l'}(\bar{k}r) - j_l(kr) j_{l'}^{(2)}(\bar{k}r) + j_l^{(2)}(kr) j_{l'}^{(2)}(\bar{k}r) \right] \right\}^2,
\end{aligned} \tag{3.32}$$

where  $r = \lambda_o - \lambda$ ,  $\bar{r} = \lambda_o - \bar{\lambda}$ ,  $s = \eta_o - \eta$ ,  $\eta_e = \lambda_e$ , and  $\eta_o = \lambda_o$ .

Next let us consider two directions with unit directional vectors  $\mathbf{e}_1$  and  $\mathbf{e}_1$ . If we define  $\mu_1, \mu_2$  and  $\beta$  as  $\mu_1 = \mathbf{k}\mathbf{e}_1/k$ ,  $\mu_2 = \mathbf{k}\mathbf{e}_2/k$  and  $\cos \beta = \mathbf{e}_1\mathbf{e}_2$ , respectively, we have a mathematical relation

$$\int \int d\phi_k d\theta_k \sin \theta_k P_l(\mu_1) P_l(\mu_2) = \frac{4\pi}{2l+1} P_l(\cos \beta). \tag{3.33}$$

Using Eq.(3.33) it is found that the correlation between  $\Theta_{pp}$ 's in the two directions is expressed as

$$\langle \Theta_{pp}(\mathbf{e}_1) \Theta_{pp}(\mathbf{e}_2) \rangle = \frac{1}{2} \sum_l \sum_{l'} \left( B_{ll'}^I + B_{ll'}^{II} + B_{ll'}^{III} \right) P_l(\cos \beta) P_{l'}(\cos \beta). \tag{3.34}$$

Here let us expand  $P_l P_{l'}$  by  $P_n$  as

$$P_l(\cos \beta) P_{l'}(\cos \beta) = \sum_n b_{ll'n} P_n(\cos \beta). \tag{3.35}$$

The derivation of the coefficient  $b_{ll'n}$  is shown in Appendix B. Then the correlation reduces to

$$(T_0)^2 \langle \Theta_{pp}(\mathbf{e}_1) \Theta_{pp}(\mathbf{e}_2) \rangle = \sum_n \frac{2n+1}{4\pi} C_n^{(2)} P_n(\cos \beta), \tag{3.36}$$

where

$$C_n^{(2)} = \frac{4\pi}{2n+1} (T_0)^2 \sum_{l,l'} \left( B_{ll'}^I + B_{ll'}^{II} + B_{ll'}^{III} \right) b_{ll'n}. \tag{3.37}$$

The expressions for  $\langle \delta_2 T/T \rangle$ ,  $\langle (\Theta_{pp})^2 \rangle$  and  $\langle \Theta_{pp}(\mathbf{e}_1) \Theta_{pp}(\mathbf{e}_2) \rangle$  in Eqs.(3.16), (3.29) and (3.36) are our new result which will be useful to derive the second-order power spectra.

#### IV. CONCLUDING REMARKS

In this paper we derived the average value  $\langle \delta_2 T/T \rangle$  and the second-order power spectra  $C_n^{(2)}$  of CMB anisotropies due to primordial random density perturbations, which include two random variables  $\alpha(\mathbf{k})$  and  $\alpha(\bar{\mathbf{k}})$ , using the average values of the products of  $\alpha(\mathbf{k})$ . The average value  $\langle \delta_2 T/T \rangle$  does not vanish and has no angular dependence. It should be regarded as the monopole component of temperature fluctuations and the second-order angular correlation is described by the power spectra  $C_n^{(2)}$ . Since we have not derived  $\langle \delta_1 T/T \delta_3 T/T \rangle$  yet, our analysis of second-order power spectra is incomplete. But we think  $\langle \Theta_{pp}^2 \rangle$  may represent the essential feature of second-order power spectra, that is, their  $l$ -dependence. In the next step we will analyze  $\langle \delta_1 T/T \delta_3 T/T \rangle$  with the third-order metric perturbations for completeness.

In the case  $\Lambda = 0$ , the first-order temperature fluctuations due to the integrated Sachs-Wolfe effect vanish, while the second-order ones do not vanish and play a dominant role in the integrated Sachs-Wolfe effect, which should not be neglected. In the cosmological local void model (LVM)[18, 19, 20, 21], the exterior region is described in terms of the Einstein-de Sitter model, and so the main part of the integrated Sachs-Wolfe effect in LVM is of second-order. In the interior region we use open low-density models with  $\Omega_0 < 1$ , in which we have nonzero first-order Sachs-Wolfe effect. It is important for showing the observational reality of LVM to derive them.



In the case  $\Lambda \neq 0$ , the first-order temperature fluctuations due to the integrated Sachs-Wolfe effect decrease rapidly with the increase of the redshift  $z$ , but the second-order temperature fluctuations due to the integrated Sachs-Wolfe effect decrease more slowly, and so the latter fluctuations may be dominant over the first-order fluctuations at the early stage. This situation is explained in a separate paper[25] using a simple model of density perturbations. Thus a characteristic behavior of CMB power spectra due to second-order temperature fluctuations will be measured through the future precise observation and bring useful informations on the structure and evolution of our universe in the future.

### Acknowledgments

The author thanks K.T. Inoue and referees for helpful discussions and comments.

### APPENDIX A: FIRST EXPRESSIONS OF $A_{ll'mm'}^I$ , $A_{ll'mm'}^{II}$ AND $A_{ll'mm'}^{III}$

$$\begin{aligned}
A_{ll'mm'}^I &= \frac{2}{8^2} (2\pi)^{-2} (-i)^{(l+l')} i^{(m+m')} (2l+1)(2l'+1)(2m+1)(2m'+1) \int \int dk d\bar{k} (k\bar{k})^2 d\mu d\bar{\mu} \mathcal{P}_F(k) \mathcal{P}_F(\bar{k}) \\
&\times \int_{\lambda_o}^{\lambda_e} d\lambda \left\{ \int_{\lambda_o}^{\lambda_e} d\bar{\lambda} P'(\lambda) P'(\bar{\lambda}) (k\bar{k})^2 (\mu\bar{\mu})^2 j_l(kr) j_{l'}(\bar{k}\bar{r}) P_l(\mu) P_{l'}(\bar{\mu}) + \left[ P'(\lambda) \left( \frac{5}{2} k\bar{k} \mu\bar{\mu} + k^2 \mu^2 + \bar{k}^2 \bar{\mu}^2 \right) \right. \right. \\
&+ \left. \frac{1}{7} P(\lambda) P'(\lambda) (k\bar{k})^2 \left( 2(\mu\bar{\mu})^2 - 6(\mu^2 + \bar{\mu}^2) + 3 - \frac{3}{2}(1 - \mu^2)(1 - \bar{\mu}^2) \right) \right] \\
&\times j_l(kr) j_{l'}(\bar{k}\bar{r}) P_l(\mu) P_{l'}(\bar{\mu}) + 2P''(\lambda) \int_{\lambda_o}^{\lambda} d\bar{\lambda} P(\bar{\lambda}) (k\bar{k} \mu\bar{\mu})^2 j_l(kr) j_{l'}(\bar{k}\bar{r}) P_l(\mu) P_{l'}(\bar{\mu}) \Big\} \\
&\times \int_{\eta_o}^{\eta_e} d\eta \left\{ \int_{\eta_o}^{\eta_e} d\bar{\eta} P'(\eta) P'(\bar{\eta}) (k\bar{k})^2 (\mu\bar{\mu})^2 j_m(ks) j_{m'}(\bar{k}\bar{s}) P_m(\mu) P_{m'}(\bar{\mu}) + \left[ P'(\eta) \left( \frac{5}{2} k\bar{k} \mu\bar{\mu} + k^2 \mu^2 + \bar{k}^2 \bar{\mu}^2 \right) \right. \right. \\
&+ \left. \frac{1}{7} P(\eta) P'(\eta) (k\bar{k})^2 \left( 2(\mu\bar{\mu})^2 - 6(\mu^2 + \bar{\mu}^2) + 3 - \frac{3}{2}(1 - \mu^2)(1 - \bar{\mu}^2) \right) \right] \\
&\times j_m(ks) j_{m'}(\bar{k}\bar{s}) P_m(\mu) P_{m'}(\bar{\mu}) + 2P''(\eta) \int_{\eta_o}^{\eta} d\bar{\eta} P(\bar{\eta}) (k\bar{k} \mu\bar{\mu})^2 j_m(ks) j_{m'}(\bar{k}\bar{s}) P_m(\mu) P_{m'}(\bar{\mu}) \Big\}, \quad (A1)
\end{aligned}$$

$$\begin{aligned}
A_{ll'mm'}^{II} &= \frac{2}{16^2} (2\pi)^{-2} (-i)^{(l+l')} i^{(m+m')} (2l+1)(2l'+1)(2m+1)(2m'+1) \int \int dk d\bar{k} (k\bar{k})^4 d\mu d\bar{\mu} \mathcal{P}_F(k) \mathcal{P}_F(\bar{k}) \\
&\times \int_{\lambda_o}^{\lambda_e} d\lambda P'(\lambda) \left[ 1 - \frac{26}{7} P(\lambda) k\bar{k} \mu\bar{\mu} \right] j_l(kr) j_{l'}(\bar{k}\bar{r}) (1 - \mu^2) P_l(\mu) P_{l'}(\bar{\mu}) \\
&\times \int_{\eta_o}^{\eta_e} d\eta P'(\eta) \left[ 1 - \frac{26}{7} P(\eta) k\bar{k} \mu\bar{\mu} \right] j_m(ks) j_{m'}(\bar{k}\bar{s}) (1 - \mu^2) P_m(\mu) P_{m'}(\bar{\mu}), \quad (A2)
\end{aligned}$$

$$\begin{aligned}
A_{ll'mm'}^{III} &= 2 \left( \frac{3}{112} \right)^2 (2\pi)^{-2} (-i)^{(l+l')} i^{(m+m')} (2l+1)(2l'+1)(2m+1)(2m'+1) \int \int dk d\bar{k} (k\bar{k})^6 d\mu d\bar{\mu} \mathcal{P}_F(k) \mathcal{P}_F(\bar{k}) \\
&\times \int_{\lambda_o}^{\lambda_e} d\lambda P(\lambda) P'(\lambda) j_l(kr) j_{l'}(\bar{k}\bar{r}) (1 - \mu^2)(1 - \bar{\mu}^2) P_l(\mu) P_{l'}(\bar{\mu}) \\
&\times \int_{\eta_o}^{\eta_e} d\eta P(\eta) P'(\eta) j_m(ks) j_{m'}(\bar{k}\bar{s}) (1 - \mu^2)(1 - \bar{\mu}^2) P_m(\mu) P_{m'}(\bar{\mu}), \quad (A3)
\end{aligned}$$

where  $r = |\mathbf{x}|$ ,  $s = |\mathbf{y}|$ ,  $r = \lambda_o - \lambda$ ,  $\bar{r} = \lambda_o - \bar{\lambda}$ ,  $s = \eta_o - \eta$ ,  $\bar{s} = \eta_o - \bar{\eta}$ ,  $\eta_e = \lambda_e$ , and  $\eta_o = \lambda_o$ .

### APPENDIX B: DERIVATION OF $b_{ll'n}$

Since the Legendre function  $P_m(z)$  is a finite power series of  $z^j$  with integers  $j(\leq m)$ ,  $P_m(z)P_{m'}(z)$  can be also expressed as a finite power series of  $z^j$  with integers  $r$  satisfying  $0 \leq j \leq m + m'$ . On the other hand,  $z^n$

is expressed as a finite series of  $P_m(z)$  with  $0 \leq m \leq n$ , so that  $P_m(z)P_{m'}(z)$  can be expressed as a finite series of  $P_n(z)$  with  $0 \leq n \leq m + m'$ . Their examples are

$$\begin{aligned} P_1 P_2 &= \frac{2}{5} P_1 + \frac{3}{5} P_3, \\ (P_2)^2 &= \frac{1}{5} P_0 + \frac{2}{7} P_2 + \frac{18}{35} P_4, \\ P_1 P_3 &= \frac{3}{7} P_2 + \frac{4}{7} P_4, \\ P_2 P_3 &= \frac{9}{35} P_1 + \frac{4}{15} P_3 + \frac{10}{21} P_5. \end{aligned} \quad (\text{B1})$$

In these cases the coefficients  $b_{ll'n}$  in Eq.(3.35) are

$$\begin{aligned} b_{121} &= 2/5, & b_{123} &= 3/5, \\ b_{220} &= 1/5, & b_{222} &= 2/7, & b_{224} &= 18/35, \\ b_{132} &= 3/7, & b_{134} &= 4/7, \\ b_{231} &= 9/35, & b_{233} &= 4/15, & b_{235} &= 10/21. \end{aligned} \quad (\text{B2})$$

The general formula of these series are classified into the following three cases.

(1) even-even type

$$\begin{aligned} P_{2m} P_{2m'} &= \sum_{s=0}^{m+m'} b_{2m \ 2m' \ 2s} P_{2s}, \\ b_{2m \ 2m' \ 2s} &= \frac{4s+1}{2^{2m+2m'}} \sum_{j=0}^m \sum_{j'=0}^{m'} (-1)^{m+m'+j+j'} (2j+2j')! (2m+2j)! (2m'+2j')! \\ &\quad / \left[ (m+j)!(m-j)!(2j)!(m'+j')!(m'-j')!(2j')!(2j+2j'-2s)!!(2j+2j'+2s+1)!! \right], \end{aligned} \quad (\text{B3})$$

(2) odd-odd type

$$\begin{aligned} P_{2m+1} P_{2m'+1} &= \sum_{s=0}^{m+m'+1} b_{2m+1 \ 2m'+1 \ 2s} P_{2s}, \\ b_{2m+1 \ 2m'+1 \ 2s} &= \frac{4s+1}{2^{2m+2m'+2}} \sum_{j=0}^m \sum_{j'=0}^{m'} (-1)^{m+m'+j+j'} (2j+2j'+2)! (2m+2j+2)! (2m'+2j'+2)! \\ &\quad / \left[ (m+j+1)!(m-j)!(2j+1)!(m'+j'+1)!(m'-j')!(2j'+1)! \right. \\ &\quad \times \left. (2j+2j'-2s+2)!!(2j+2j'+2s+3)!! \right], \end{aligned} \quad (\text{B4})$$

(3) even-odd type

$$\begin{aligned} P_{2m} P_{2m'+1} &= \sum_{s=0}^{m+m'} b_{2m \ 2m'+1 \ 2s+1} P_{2s+1}, \\ b_{2m \ 2m'+1 \ 2s+1} &= \frac{4s+3}{2^{2m+2m'+1}} \sum_{j=0}^m \sum_{j'=0}^{m'} (-1)^{m+m'+j+j'} (2j+2j'+1)! (2m+2j)! (2m'+2j'+2)! \\ &\quad / \left[ (m+j)!(m-j)!(2j)!(m'+j'+1)!(m'-j')!(2j'+1)! \right. \\ &\quad \times \left. (2j+2j'-2s)!!(2j+2j'+2s+3)!! \right], \end{aligned} \quad (\text{B5})$$

where the rules of ! and !! are

$$\begin{aligned} n! &= 1 \cdot 2 \cdots (n-1) \cdot n & \text{for } n \geq 1, \\ 0! &= 1, \\ n! &= \infty & \text{for } n < 0, \end{aligned} \quad (\text{B6})$$

and

$$(2n)!! = 2 \cdot 4 \cdots (2n-2) \cdot 2n \quad \text{for } n \geq 1,$$

$$\begin{aligned}
(2n+1)!! &= 1 \cdot 3 \cdots (2n-1) \cdot 2n+1 & \text{for } n \geq 0, \\
0!! &= 1, \\
n!! &= \infty & \text{for } n < 0.
\end{aligned}
\tag{B7}$$

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- [1] C.L. Bennet et al., *Astrophys. J. Suppl.* **148**, 1 (2003).
  - [2] D.N. Spergel et al., *Astrophys. J. Suppl.* **148**, 175 (2003).
  - [3] E. Komatsu et al., *Astrophys. J. Suppl.* **148**, 119 (2003).
  - [4] S. Mollerach, A. Gangui, F. Lucchin and S. Matarrese, *Astrophys. J.* **453**, 1 (1995).
  - [5] D. Munshi, T. Souradeep and A. Starobinsky, *Astrophys. J.* **454**, 552 (1995).
  - [6] D.N. Spergel and D.M. Goldberg, *Phys. Rev.* **D59**, 103001 (1991).
  - [7] N. Bartolo, S. Mollerach and A. Riotto, *JCAP* **05**, 010 (2006).
  - [8] N. Bartolo, S. Mollerach and A. Riotto, *JCAP* **06**, 024 (2006).
  - [9] K. Tomita, *Phys. Rev.* **D71**, 083504 (2005). See also K. Tomita, *Prog. Theor. Phys.* **37**, 831 (1967), H. Russ, et al., *Phys. Rev.* **D53**, 6881 (1996), and S. Matarrese, et al., *Phys. Rev.* **D58**, 043504 (1998).
  - [10] S. Mollerach and S. Matarrese, *Phys. Rev.* **D56**, 4494 (1997).
  - [11] K. Tomita, *Phys. Rev.* **D72**, 043526 (2005).
  - [12] K. Tomita, *Phys. Rev.* **D72**, 103506 (2005).
  - [13] K. Tomita, *Phys. Rev.* **D73**, 029901(E) (2006).
  - [14] K.T. Inoue and J. Silk, *Astrophys. J.* **648**, 23 (2006).
  - [15] K.T. Inoue and J. Silk, *Astrophys. J.* **664**, 650 (2007).
  - [16] G. D'Amico et al., arxiv: 0707.2894.
  - [17] J. Hwang and H. Noh, *Phys. Rev.* **D72**, 044012 (2005).
  - [18] K. Tomita, *Astrophys. J.* **529**, 26 (2000); K. Tomita, *Astrophys. J.* **529**, 38 (2000); K. Tomita, *MNRAS* **326**, 287 (2001); K. Tomita, *Prog. Theor. Phys.* **106**, 929 (2001).
  - [19] M.N. C  l  rier, *Astron. Astrophys.* **353**, 63 (2000); M.N. C  l  rier, *New Adv. Phys.* **1**, 29 (2007).
  - [20] H. Alnes, M. Amarzguoui and O. Gron, *Phys. Rev.* **D73**, 083519 (2006).
  - [21] T. Biswas, R. Mansouri and A. Notari, astro-ph/0606703.
  - [22] N. Sugiyama, *Astrophys. J. Suppl.* **100**, 281 (1995).
  - [23] M. Zaldarriaga, *Large-Scale Structure Formation* (Kluwer Academic Publ., Netherlands, 2000), p 213.
  - [24] W. Hu and M. White, *Phys. Rev.* **D56**, 596 (1997).
  - [25] K. Tomita and K.T. Inoue, astro-ph/0712.1291, to be published in *Phys. Rev. D*.